Virtual Skies

Air Traffic Management

Activity I: Determining the Locus of a Flight Plan

Introduction

As soon as a flight path is decided upon, the pilot and controllers monitor the course of the airplane to be sure the airplane stays on that course. This is not unlike making sure a car stays on a road (and between the lines). If we describe a flight path in terms of a locus defined by lines with equations, it should be easy to determine if we are inside the locus and if the actual airplane's path intersects any of the locus' boundaries (or lines). Controllers have programs that do essentially this, and they can foresee when an airplane may leave this locus. To better understand how such programs work, let's look at a flight path specifically in terms of objective points, then regions (paths).

This is broken up into eight sections:

Part A. At what point will the airplane be after time, t?
Part B. What is the distance traveled in time t?
Part C. Can we reach a specific endpoint if given v_x and v_y are constants?
Part D. If the airplane does not reach (x_1, y_1), what is the closest point of approach (x_2, y_2) where it should turn to get to (x_1, y_1)?
Part E. If we want to reach (x_1, y_1), what should the heading angle (x) be?
Part F. If I want to reach (x_1, y_1) at $t = t_f$, what should I do?
Part G. If the airplane flies for a certain speed for a certain amount of time, at what speed should the airplane fly for the remaining portion of time, in order to still reach its destination at time t_f?
Part H. Staying on the geometric path

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Part A - Where will the airplane be after time t?

1. What I already know:
   a) d = ___________ x ____________, where
   d= distance, _____ = ________________, and ______ = _______________
   This means, if I start from the origin (0,0) and travel at 40 mi/h for 2 hours, I will have traveled ___________________ miles.
   If I then travel for another hour at 20 mi/h, I will have travelled another ______________________ miles.
   The total trip will be _____________________________ miles because total distance traveled = sum of all parts of trip.

   b) Not all distances are known or can be easily calculated for all problems. Sometimes, we have to leave things in variable form. If you were to write an equation with variables to show that the total distance = the sum of all distances, what would the equation look like? Use d and subscripts to denote different distances.

   If you don't know the distance of the first stretch of the path, and only want to use the r and t variables for that path, what would the new equation look like?

   What is another way of writing this equation, using no ds?

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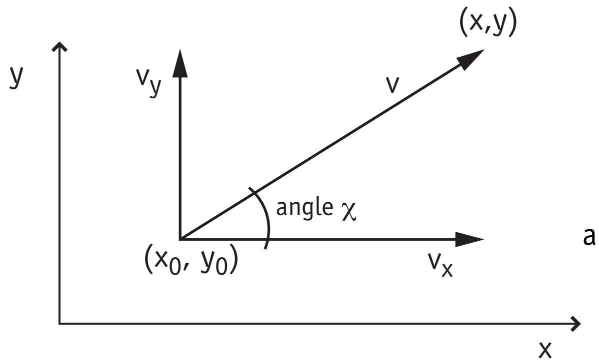
c) Our problem becomes a little more complex when we use x and y to illustrate points and we travel along diagonals. We can use the following variables to determine where an airplane will be (x, y) after time t.



angle χ = heading angle

Similarly as with distance, the final position = where the pilot began from (x_0,y_0) plus the new distance traveled. For convenience, the distance vector is broken up into its x and y constituents.

$$x = x_0 + v_x t$$
$$y = y_0 + v_y t$$

2. Solve for the unknown variables; fill in the chart.

	x_0	y_0	v_x	v_y	t	x	y
1.	0	0			5	15	10
2.	2	2	60	25	3		
3.	4	1	-10	30	5		
4.			-15	-15	2	-25	-40
5.	-10	2	17	20		24	42

3. For any two of the above situations, summarize the path of the airplane as the example shows, on a coordinate grid. Estimate the distance traveled using a ruler or the squares on your graph paper, or use the distance formula if you know it.

Part B - What is the distance traveled in time t?

Let's derive a formula then use it to solve for distance with respect to time!

1. First method of derivation: Proof with Algebra and Trigonometry Properties.

Given: distance is defined with the following equation:

$$d = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$$

and

$x = x_0 + v_x t$

$y = y_0 + v_y t$ (and variables defined in picture in Part A)

Prove: $d = vt$

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	Statements	Reasons	Reference Numbers
1.		Given	- - - - - - -
2.	$x - x_0 = v_x t$ $y - y_0 = v_y t$		(1)
3.		Given	- - - - - - -
4.	$d = ((vxt)^2 + (vyt)^2)^{1/2}$		
5.	$d = t\,(v_x^{\,2} + v_y^{\,2})^{1/2}$		
6.	$\cos\chi = v_x / v$ and	Definition of cosine and sine, given figure	
7.		Multiplication Property of Equality	(6)
8.	$v_x + v_y = v\cos\chi + v\sin\chi$		(7)
9.		Distributive Property	(8)
10.	$v_x^{\,2} + v_y^{\,2} = v^2(\cos^2\chi + \sin^2\chi)$		(9)
11.	$v_x^{\,2} + v_y^{\,2} = v^2(1)$		(10)
12.		Substitution Property	(5, 11)
13.	$d = tv$		(12)

2. Use calculus to derive the same formula.

δ is delta, or change.

Given: Definition of velocity = rate of change in position or change in distance over change in time: $\delta d/\delta t = v$,

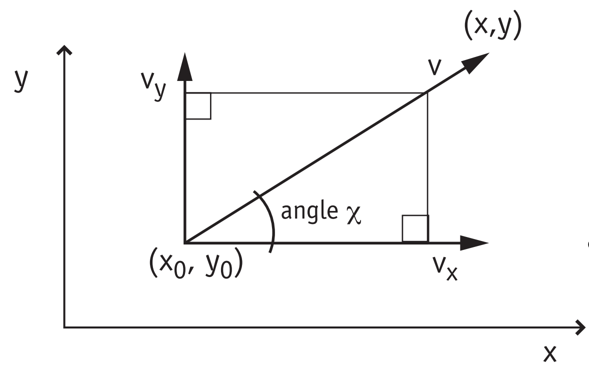
Prove: $d = vt$

	Statements	Reason/Explanation	Reference Numbers
1.			- - - - - - -
2.	$\delta d = v\delta t$		
3.	$\delta d = \quad vdt$	take integral of both sides	(2)
4.		solving, definition of integral	
5.		at t = 0 and d = 0 (initial conditions)	

3. We can solve for distance using d = vt.

To do this, we must use a single vector, as opposed to the vector components, v_x, and v_y.

a) How could you use the vector components to find the length of v? The following drawing should help you.:



angle χ = heading angle

b) Vectors can be negative, as well as positive.

i) If a vector was negative, what would that mean? Draw an example of a situation where a vector is negative.

ii) Which of the above methods of solving provides no negative vectors. Explain why no negative vectors are included in the solution for v.

iii) When would the other method provide a negative v vector?

c) Solve for the unknown in the following chart, using Pythagorean Theorem and d = tv.

	v_x	v_y	v	t	d
1.	5	3		5	
2.	4	7			$(9360)^{1/2}$
3.		6	9	3.5	
4.	$3(7)^{1/2}$		12		90

Draw any two situations from the chart on graph paper, using coordinate geometry, and estimate the variables by using a ruler and/or the distance formula. Check your answers against the estimates - do they make sense?

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d) Solve for the unknowns, using trigonometry and d = vt.

	Heading Angle (degrees)	v_x or v_y	v	t	d
1.	19°	$v_x = 12$		3	
2.	78°	$v_y = 20$		4	
3.	45°		27	6	162
4.	30°		30	2	60
5.	-15°		-25	6	-150

Draw any two of the situations from question 3 on graph paper with a protractor, and the variables using a ruler. Check your answers against the estimates - do they make sense? What is an important thing about negative values in these problems that must be realized so you do not erroneously say there is "no solution" ?

Part C - Can we reach a specific endpoint if given v_x and v_y are constants?

In other words, will (x_1,y_1) be a point in our path, if v_x and v_y are constants?

1. Draw the two possible situations here : (x_1,y_1) is in the locus of the path, or it is not.

2. How could you distinguish algebraically between the two options above?

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3. Let's write an equation for a line, using a) what we know about lines, and b) what we know already about the relationships between variables.

What is the equation of a line, which is easy to plug possible points into? Give the general form of the equation and then show an example of plugging a point into it.

General Form: **Example:**

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4. Let's come up with a formula that uses facts we already know. That is, given:
$x = x_0 + v_x t$ and $y = y_0 + v_y t$,
create an equation for a line that is in one of the above forms.

Go through this process, below, and fill in the blanks.

Statements	Reasons	Reference No.s
1. $x = x_0 + v_x t$	Given	- - - - - -
2. $x - x_0 = v_x t$		
3. $(x - x_0) / v_x = t$		
	Given	- - - - - -
	Substitution	
6. $y = y_0 + (v_y / v_x) x - (v_y / v_x) x_0$		(5)
OR		
	Subtraction Property of Equality (in Point-Slope form)	(5)
	Reorganization of Terms (in Slope-Intercept form)	(6)

5. Before you solve some equations, determine which situations would be ideal for using each equation.
   a) If given: **everything except v_y** it is best to use the ___________________ equation.
   b) If given: **everything but y** it is best to use the _____________________ equation.
   c) If given: **everything but y_0** it is best to use the ____________________ equation.
   d) If given: **everything but v_x** it is best to use the ____________________ equation.
   e) If given: **everything but x or x_0**, it is best to use the ___________________ equation.

6. Will the following planes reach their destinations?

	x_0	y_0	v_x	v_y	x	y	Reach (x_1, y_1)? Yes/No
1.	2	5	6		12	28	yes
2.	1	-7	25	32	19		yes
3.	0	9		-2	8	-7	yes
4.		25	-5	-4	11	9	yes
5.	7	9	2	3	19	13	
6.	14	0	5	-5	0	14	
7.	3	-5	-12	40	9	-25	
8.	7	-2.5	11	8	18	5	
9.	-6	3.125	132	47.5	27	15	
10.	0.5	0.67	43	68	7.67	12	

7. Draw four of the above situations on graph paper. Use the vectors to draw the path from (x_0,y_0). Be sure your drawing agrees with your answers above!

Part D - If the airplane does not reach (x_1, y_1), what is the closest point of approach (x_2, y_2) where it should turn to get to (x_1, y_1)?

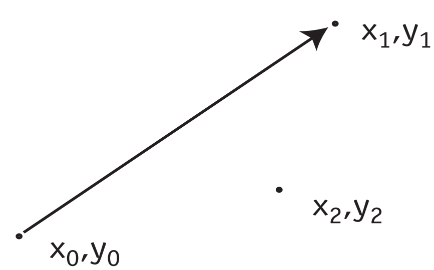
1. Based on what you know from geometry, what is the measure of the angle for turning such that the path between the original destination (x_2, y_2) and (x_1, y_1) is shortest?

   __________________ degrees

2. On graph paper, redraw one of the examples from Part C where the airplane did not arrive at (x_1,y_1). Add (x_2, y_2) and the measure of the angle of turning in the drawing.

3. If (x_0,y_0), (x_1,y_1), and (x_2, y_2) serve as vertices, what kind of shape do you get (be specific)? ____

Need a hint for the answer above?
Answer this: In the graphic below, if you are travelling along the vector, where would you turn so you travel the shortest distance from the vector, to get to (x_2,y_2)?



4. Solve for the following:

   (x_2, y_2) = ________________
   distance between (x_0,y_0) and (x_1, y_1) = ____________________
   distance between (x_2,y_2) and (x_1, y_1) = ____________________
   distance between (x_0,y_0) and (x_2, y_2) = ____________________
   measures of other two angles = _________ degrees and ___________ degrees

   Label these items on your drawing.

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5. We can also find the closest point of approach using calculus. Use the partial proof below to explain how we derive a formula for this, using calculus.

	Statements	Reasons	Reference No.s
1.		given	- - - - -
2.	$\frac{\delta (d_2^{\,2})}{(\delta x_2)} = 2(x_2 - x_1)(\delta x_2/\delta x_2) + 2(y_2 - y_1)(\delta y_2/\delta x_2)$		(1)
3.		Condition for Extremum (because we are interested in the shortest distance between two points)	- - - - -
4.		Velocity Vectors	- - - - -
5.		Substitution	(2, 3, 4)
6.		Given. Equation for line between (x_0, y_0) and (x_2, y_2). (See equation derived in Section C)	- - - - -
7.	$0 = x_2 - x_1 + (v_y/v_x)[(v_y/v_x)x_2 + (y_0 - (v_y/v_x)x_0)] - (v_y/v_x)y_1$		
8.	$0 = x_2[1 + (v_y/v_x)2 - x_1 + (v_y/v_x)(y_0 - v_y/v_x)x_0] - (v_y/v_x)y_1$		(7)
9.		Division Property of Equality	(8)

As soon as we have solved for x_2, we can substitute this value into our equation for our line containing (x_0, y_0) and (x_2, y_2), to find y_2.

6. Use your values from questions 1 to 4 to solve for x_2 and y_2.

Part E. If we want to reach (x_1, y_1), what should the heading angle (c) be?

1. How could we solve this problem?

2. We can write a general formula to find the heading angle for use in any situation. Fill in the proof below.

	Statements	Reasons	Reference No.s
1.	$v_x = v \cos \chi$ $v_y = v \sin \chi$		- - - - -
2.	$(y_1 - y_0) = (v_y / v_x)(x_1 - x_0)$		- - - - -
3.	$(y_1 - y_0) = \dfrac{v \sin \chi}{v \cos \chi}(x_1 - x_0)$		
4.			
5.	$y_1 - y_0 = \tan \chi \, (x_1 - x_0)$		
6.	$\tan \chi = \dfrac{(y_1 - y_0)}{(x_1 - x_0)}$		
7.			

3. Solve for the heading angle, below, to the nearest hundredth degree.

	χ	y_1	y_0	x_1	x_0
1.		-3	0	1	0
2.		17	10	12	5
3.		-18	6	-25	2
4.		-12	-2	5	15
5.		25	4	-40	-10

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4. Draw each of the five situations from the table in #3. Do the heading angles agree with your drawings?

5. Now answer these questions with respect to your drawings and the chart in #3.

a) There are situations in #3 which might be better explained using other angles. What would these be?

b) If you wanted to warn a fellow student to "look out" for these situations in the future, or you wanted to generalize when you might have to change the way you write the degrees (as in Part B), what would you say? *Hint: Think about starting and ending points and their a) sign, b) relative magnitude, c) graphs you drew.*

Part F - If I want to reach (x_1, y_1) at $t = t_f$, what should I do?

This, of course, is one of the most popular questions in travel! Can you think of situations where this has been important?

1. If we would like to calculate time, given specific speeds and distances, we will need to alter a few equations that we already know. Fill in the simple proof, below.

	Statements	Reasons	Reference No.s
1.		Given : distance	- - - - -
2.		Distance Formula	- - - - -
3.	$(v)(t) = [(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}$		(1, 2)
4.		Division Property of Equality	(3)

Because we are only interested in t_f, our equation is:

2. From this equation, we can see that time is a function of

____

____

and ____________________________ .

3. Fill in the following chart, solving for t_f.

	Where I Was	Destination	Velocity	Final Time
1.	(4, 3)	(12, 15)	4	
2.	(5, -7)	(-18, 21)	7	
3.	(-2, 9)	(-11, 17)	-3	
4.	(-6, -8)	(16, -10)	$-4\ (2)^{1/2}$	

4. What does it mean to have a negative final time? In what kind of situations does this occur? Can you think of an example? How would this situation look, graphically?

5. For the previous equation (from the proof), the known variables are (x_0,y_0) (x_1,y_2), and t_f. If we want to solve for v, we should rewrite our equation. In other words, we will need to choose or change a specific velocity in order to reach (x_1,y_1) in a specific time period.

   a) Our new equation is:

   v =

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b) Let's apply this notion to a trip to grandmother's house, which is in Mulino, Oregon, at latitude and longitude point (123, 45). In the following chart, fill in the following required speeds in order to get to her house from the following initial locations, with the given final times (t_f).

Also determine what the starting points are, using a globe!

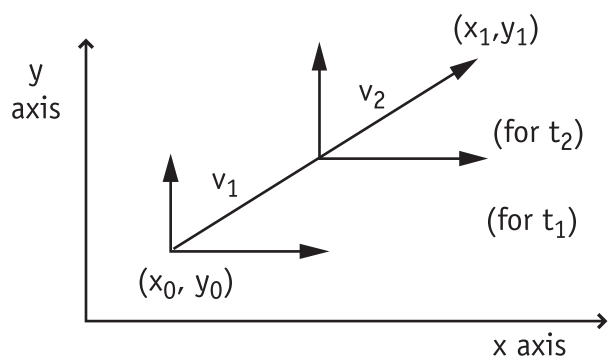
	y_0	x_0	t_f	Starting Point (approx)	v
1.	45	107	5		
2.	20	160	$(997)^{1/2}$		
3.	37	122	$(13)^{1/2}$		
4.	43	0	$10(37)^{1/2}$		
5.	52	130	14		

Part G - If the airplane flies for a certain speed for a certain amount of time, at what speed should the airplane fly for the remaining portion of time, in order to still reach its destination at time t_f?

The calculations from the preceding sections are certainly useful to pilots, for determining how fast they should attempt to travel, in order to get somewhere in a specified amount of time. However, because flying speeds are usually predetermined, it is more likely that final time will be adjusted so that an airplane is not travelling at dangerously fast or slow speeds.

A more realistic situation would be if an airplane had to adjust its speed in order to reach a specific place in a certain amount of time, because weather or controller instructions had forced it to slow down for a period of time.

A picture helps explain this situation and all of the contributing factors.:



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1. The total distance traveled is:
   $d =$ ____________________

2. The distance already flown (d_1) is:
   $d_1 =$ ____________________

3. The distance remaining (d_2) is the total distance minus the distance already flown.
   $d_2 =$ ____________________

4. The remaining time (t_2) is the final time minus the time already flown.
   $t_2 =$ ________ - ________

5. We know that $v = d/t$.
   So v_2 = distance remaining / remaining time
   or $v_2 = d_2 / t_2$.
   Rewrite this equation using known terms only; do not use t_2 and d_2.

   $v_2 =$ ____________________

6. Multiply both sides of the equation by the denominator so that there are no more fractions to deal with!
   v_2 (   ) = ____________________

7. Move extra terms to the left so you can solve for distance (in terms of x and y values).
   v_2 (   ) +   = ____________________

8. We know that
   $[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2} = vt_f$,
   so

   v_2 (   ) + $= vt_f$

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9. If we divide both sides by t_f, we can solve for velocity.

$$v = \frac{\qquad}{\qquad}$$

10. If we attempt to make the denominator as similar to the numerator as possible, we get

$$v = \frac{v_2(t_f - t_1) + v_1 t_1}{(t_f - t_1) + t_1}$$

or

$$v = v_2 t_2 + v_1 t_1 / t_f$$

or

average speed = distance of one part + distance of other part/total time

This means that you will reach (x_1, y_1) at time t_f, as long as your average speed is v.

Using this information, fill in the following chart:

	t_1	v_1	t_f	v_2	v
1.	5	1	10	2	
2.	5	1	20	2	
3.	5	3	17	5	
4.	2	7	10	8	
5	1	12	15	6	

Part H - Staying on the geometric path

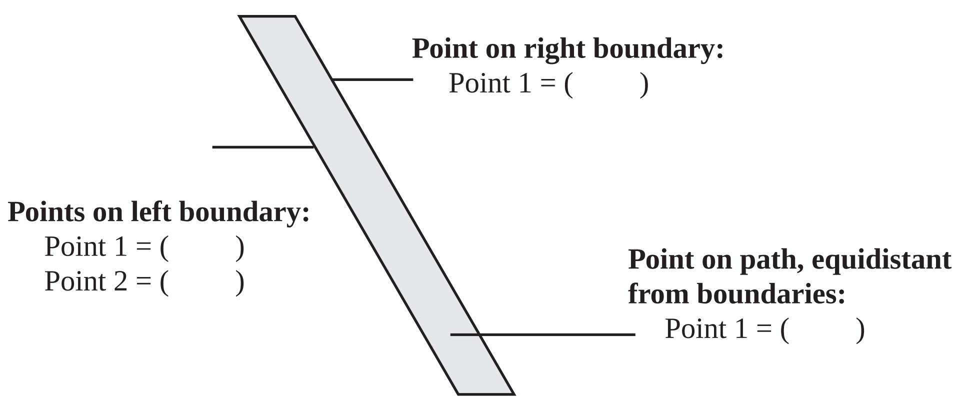
As soon as a flight path is decided upon, the pilot and controllers monitor the course of the airplane to be sure the airplane stays on that course. This is similar to making sure a car stays on a road (and between the lines).

If we describe a flight path in terms of a locus defined by lines with equations, it should be easy to determine if we are inside the locus and if the actual airplane's path intersects any of the locus' boundaries (or lines).

Controllers have programs that do essentially this, and they can foresee when an airplane may leave this locus. To better understand how such programs work, let's look at a flight path specifically in terms of objective points, then regions (paths).

Let us assume that the following are paths that an airplane must adhere to. Use a piece of thin graph paper to trace the images (being sure to think of the easiest way to trace - hint: (0,0). Pick the required points on each path and write the specified number of equations for the path.

1. For a straight path with parallel boundaries, determine the equations necessary to define the left and right boundaries, and the ideal path taken by the airplane (equidistant from the boundaries of the path).

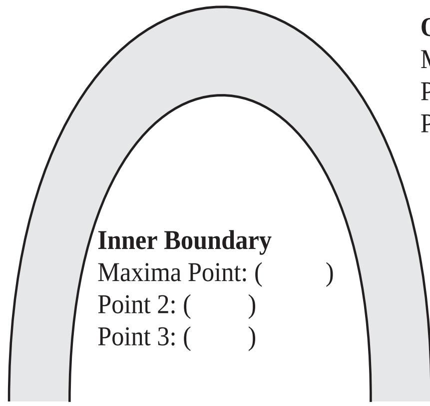


Equation for left boundary: y =

Equation for right boundary: y =

Equation for path: y =

2.



Outer Boundary
Maxima Point: (   )
Point 2: (   )
Point 3: (   )

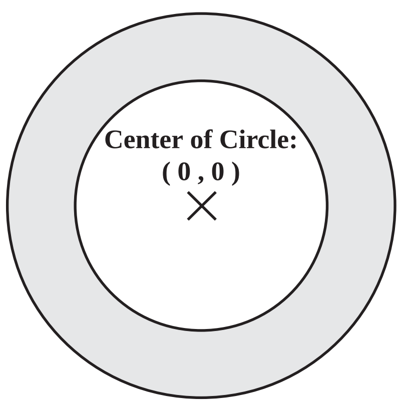
On Path
Maxima Point: (   )
Point 2: (   )
Point 3: (   )

Equation for outer boundary:

Equation for inner boundary:

Equation for path:

3.



Outer Circle
Point 1: (   )

On Path
Point 1: (   )

Inner Circle
Point 1: (   )

Equation for outer circle:

Equation for inner circle:

Equation for path:

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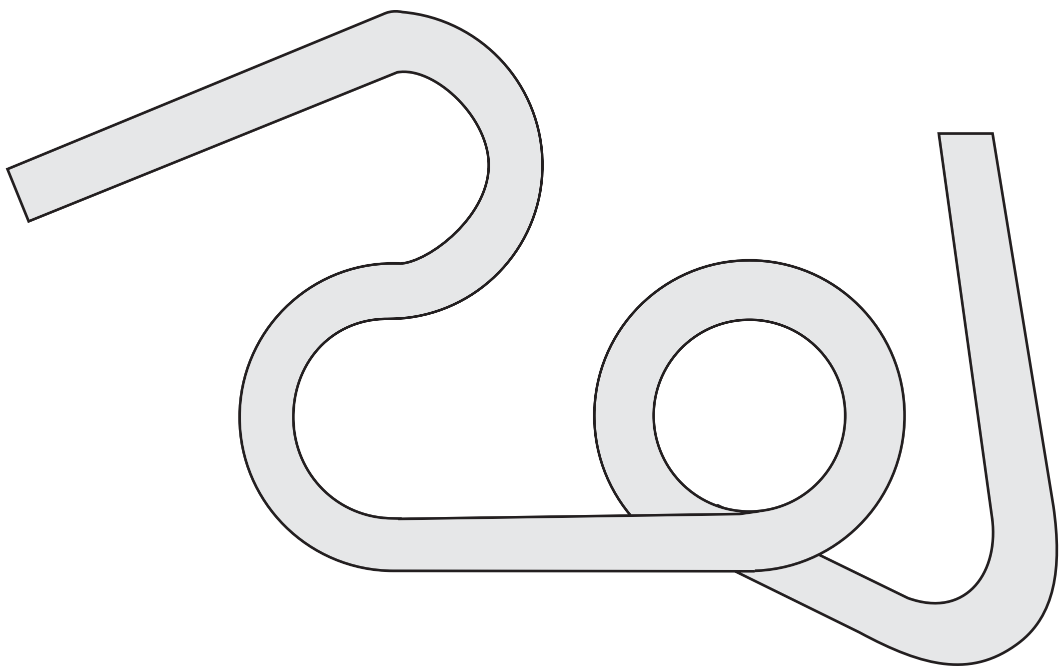
4. Use what you have practiced to describe mathematically this path.
   Hint: Trace the path onto graph paper, then determine the equations for each component. You may re-set coordinates or use one common coordinate system. You may also draw each component separately on different grids, to determine equations.



5. Instead of simply describing the complex path in terms of equations of lines, parabolas, or circles, it is important to include information about distances. If we combine the slope information, intercept information, and distance, we are providing the same information as that which is found in a vector.

   So let's use vectors!
   Use vectors to describe the path above. For complex figures like circles and parabolas, break the "rounded" figures up into multiple "straight" figures. Your teacher may give you instructions on how close your "straight" model is to the curved one - be sure to ask!